Zero Modes for the $D\!=\!11$ Membrane and Five-brane

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Abstract

There exist extremal p-brane solutions of $D\!=\!11$ supergravity for p=2 and 5. In this paper we investigate the zero modes of the membrane and the five-brane solutions as a first step toward understanding the full quantum theory of these objects. It is found that both solutions possess the correct number of normalizable zero modes dictated by supersymmetry.

I. INTRODUCTION

Recent discussions of dualities in string theory indicate that $D = 11$ supergravity may be relevant to string theory. For example, it has been argued [1–3] that $D = 11$ supergravity is the strong coupling limit of $D = 10$ Type IIA string theory. Coupled with the conjecture [1–3] that $D = 10$ Type IIA strings are compactified $D = 11$ membranes [4], it follows that $D = 11$ supergravity is relevant to $D = 10$ Type IIA theory [2,5]. The study of the $D = 11$ membrane and five-brane solutions are therefore of interest.

In this paper, the effective actions for the membrane and five-brane solutions of $D = 11$ supergravity are calculated from the $D = 11$ supergravity Lagrangian. In Section II we review the solutions. In Section III we discuss the zero modes and find the effective actions. In particular, we find the effective action to agree with the action expected from the general theory of p-branes. We make some observations about the results in Section IV.

II. PRELIMINARIES – THE SOLUTIONS

First we discuss the nature of the membrane and five-brane solutions [6–8]. These p-branes solve the $D = 11$ supergravity equations of motion which follow from the Lagrangian [7,9] (omitting fermions)

$$
\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}\sqrt{-g}R - \frac{1}{24}\sqrt{-g}F_{MNPQ}F^{MNPQ} \\
& + \frac{4}{(12)^4}\varepsilon^{MNOPQRSTUVW}F_{MNOP}F_{QRST}A_{UVW}.
\end{aligned}
\tag{1}
$$

The extreme membrane solution to $D=11$ supergravity can be written [7]

$$
\begin{cases}
ds^2 &= \Lambda^{-\frac{2}{3}}\eta_{\mu\nu}dx^{\mu}dx^{\nu} + \Lambda^{\frac{1}{3}}dy^{p}dy^{p} \\
A_{\mu\nu\rho} &= \pm\frac{1}{2}\varepsilon_{\mu\nu\rho}\Lambda^{-1}.
\end{cases}
\tag{2}
$$

where

$$
\Lambda = \left[1 + \left(\frac{r_h}{\rho}\right)^6\right].
\tag{3}
$$

The conventions used throughout this paper are that capital latin letters denote generic $D=11$ indices, world-brane coordinates are labelled by x^μ where $t = x^0$, and the remaining coordinates are labelled by y^m. Greek indices are raised and lowered with η, lower-case Latin indices are raised and lowered with δ, and all other metric factors are shown explicitly. The radial coordinate $\rho \equiv \sqrt{y^p y^p}$; $\eta_{\mu\nu}$ has signature $+1$; $\varepsilon^{012} = +1$, etc. Also, the four-form field F is obtained from the three-form potential A.

The causal structure of the membrane has been discussed in detail in Ref. [10]. It will only be mentioned that the Penrose diagram is (qualitatively) identical to that for the extreme Reissner-Nordström black hole. That coordinates can be found that smooth out the singularity at the horizon follows from the fact that the membrane solution has the structure of $adS_4 \times S^7$. Note that in the isotropic coordinates of equation (2), the horizon is located at $\rho = 0$ and hence the inside of the membrane cannot be examined in these coordinates. However, as is well known, complete spacelike slices exist that do not penetrate the horizon.

The extremal five-brane solution can be written as [8]

$$
\begin{cases}
ds^2 & = \Delta^{\frac{1}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + \Delta^{-2} dr^2 + r^2 d\Omega_4^2 \\
F & = q_m \boldsymbol{\epsilon}_4
\end{cases}
\tag{4}
$$

where $\boldsymbol{\epsilon}_4$ is the volume element on the unit S^4 surrounding the five-brane, Δ is given by

$$
\Delta = \left[1 - \left(\frac{r_h}{r} \right)^3 \right],
\tag{5}
$$

and q_m and r_h are related by

$$
q_m = \pm 9\, r_h{}^3.
\tag{6}
$$

This can also be put into an isotropic form given by

$$
\begin{cases}
ds^2 & = \Delta^{\frac{1}{3}}\, \eta_{\mu\nu} dx^\mu dx^\nu + \Delta^{-\frac{2}{3}}\, dy^p dy^p \\
F & = \frac{q_m}{4! \rho^5}\, \varepsilon_{pqrst}\, y^p\, dy^q \wedge dy^r \wedge dy^s \wedge dy^t
\end{cases}
\tag{7}
$$

where r and $\rho \equiv \sqrt{y^p y^p}$ are related by $r^3 = \rho^3 + r_h{}^3$. Finally, the field strength tensor, F_{ABCD}, is related to the four form, F, by

$$F = \frac{1}{4}F_{ABCD}dX^A \wedge dX^B \wedge dX^C \wedge dX^D, \tag{8}$$

where $X^A = \{x^0, \ldots, x^5, y^1, \ldots, y^5\}$.

The causal structure of this solution was investigated in [11]. It was found that the solution can be continued past the horizon in a totally nonsingular manner. The Penrose diagram is given in Fig. (1). It is clear from this diagram that there are complete spacelike slices with $r > r_h$ everywhere. The zero modes of the five-brane will be defined on such a slice.

III. ZERO MODES AND NORMALIZABILITY

In Ref. [7] it was shown that exactly half of the $D = 11$ supersymmetries are broken by the membrane solution. Similar results should hold for the five-brane solution. Since a $D = 11$ Majorana spinor has 32 real components, it follows that the supersymmetry breaking leaves 16 (normalizable) fermionic zero modes, and hence (by supersymmetry), eight (normalizable) bosonic zero modes. In the case of the membrane, these will be the eight translational zero modes. By spherical symmetry, it is reasonable to suppose that the effective Lagrangian in static gauge will simply be (to lowest order)

$$\mathcal{L}_{eff} = -N\eta^{\mu\nu}\partial_\mu\lambda^p\partial_\nu\lambda^p, \tag{9}$$

where N is a normalization constant that can be absorbed into λ^p, and is therefore meaningless at this order. It will be shown that equation (9) holds, with some arbitrariness in the value of N.

In the case of the five-brane, the situation is more complex. Here there are only five translational zero modes. The remaining three must be found elsewhere. It will be shown that these zero modes exist and are normalizable. Also, the effective Lagrangian for the translational zero modes is found to be given by an expression analogous to equation (9).

A. The membrane solution

Obviously, all infinitesimal diffeomorphisms $y^{p\prime} = y^{p\prime}(y^q)$ generate zero modes; however, only those which correspond to global translations will be normalizable. In flat space it is easy to determine what a global translation is. This is more difficult in curved space. Fortunately, the metric in equation (2) is asymptotically flat; thus global translations must approach $y^p \rightarrow y^p - \epsilon^p$ as $y^p \rightarrow \infty$, where ϵ is a constant vector orthogonal to the membrane+time. However, there is no well-defined rigid translation away from $\rho = \infty$. Specifically, as space becomes curved, the translation vector ϵ may be rotated and dilated. In fact, such diffeomorphisms must be considered in order to cut off the integral over the off-brane coordinates (y^m).

Thus the Lagrangian is expanded to second order, with

$$\delta g_{AB} = \lambda_{(i)} \pounds_{\epsilon_{(i)}} g_{AB} \tag{10}$$

and a similar equation for δA_{MNP}. Here, i labels the zero modes (which are eight in number); $\lambda_{(i)}(x^\mu)$ is the collective coordinate for the ith zero mode; the $\epsilon^p_{(i)}(y^m)$ are a set of eight linearly independent, off-brane vector fields ($\epsilon^\mu = 0$) with the property $\lim_{r \to \infty} \epsilon^m_{(i)} = \delta_{mi}$; and $\pounds_\epsilon$ denotes the Lie derivative with respect to the vector field ϵ^p. Then,

$$\delta^2 \mathcal{L} = \left(\Lambda_{,m} \epsilon^m_{(i)} \partial_n \epsilon^n_{(j)} + \Lambda \partial_m \epsilon^m_{(i)} \partial_n \epsilon^n_{(j)} - \frac{1}{2} \Lambda \partial_m \epsilon^n_{(i)} \partial_m \epsilon^n_{(j)} - \frac{1}{2} \Lambda \partial_m \epsilon^n_{(i)} \partial_n \epsilon^m_{(j)} \right) \eta^{\mu\nu} \partial_\mu \lambda_{(i)} \partial_\nu \lambda_{(j)}. \tag{11}$$

Integrating equation (11) over the off-brane coordinates gives equation (9), as expected.

B. The five-brane

Analogously to the membrane, the five-brane will have translational zero modes. In this case, there will be five of them. As eight (normalizable) bosonic zero modes are expected from supersymmetry, we must find three more zero modes elsewhere. These extra zero

modes are very similiar to those of the five-brane of type IIA String Theory investigated in [6].

The equation of motion for small fluctuations of the potential A around a constant background is

$$d(\hat{*}dA) + \frac{1}{3}F \wedge dA = 0 \tag{12}$$

where here F is the background field, A is the infinitesimal fluctuation and $\hat{*}$ is the full 11-dimensional (11D) Hodge dual. This is solved by

$$A = \Delta e^{ik_\mu x^\mu} U \wedge *F \tag{13}$$

$$= \Delta e^{ik_\mu x^\mu} U \wedge \frac{dr}{r^4} \tag{14}$$

where $*$ is the Hodge Dual on the 5D space perpendicular to the world-brane, k is a constant (anti-)self-dual null vector in two dimensions tangent to the world-brane and U is a constant (anti-)self-dual polarization tensor in the four spatial dimensions orthogonal to k and tangent to the world-brane. In order to obtain a normalizable zero mode, $U \wedge k$ must either be self-dual or anti-self-dual, depending on the sign of q_m. These zero modes lead to an (anti-)self-dual three-form on the world-brane, which contains three bosonic degrees of freedom. As mentioned above, these zero modes are very similiar to those of the five-brane of type IIA string theory.

The analysis of the translational zero modes is very similiar to the analysis of those of the membrane. Once again it is found that a suitable choice of $\epsilon^p(y)$ yields normalizable zero modes whose effective action is

$$\mathcal{L}_{eff} \propto -\eta^{\mu\nu}\lambda^p{}_{,\mu}\lambda^p{}_{,\nu}. \tag{15}$$

in static gauge.

IV. CONCLUSIONS

It has been shown that in static gauge, the effective membrane and five-brane actions are what would be expected from the general theory of p-branes. In the case of the membrane,

all eight bosonic zero modes are found to be translational, whereas, for the five-brane, five are translational and the rest come from an (anti-)self-dual three-form on the world-brane. The results for the five-brane are consistent with a duality between $D = 11$ supergravity and $D = 10$ type IIA string theory. Upon compactification of one of the off-brane dimensions, one of the translational zero modes of the five-brane becomes a zero mode of a U(1) field generated by the Kaluza-Klein mechanism. As predicted in [12], this set of four translational zero modes, three zero modes due to a self-dual antisymmetric three-form and one zero mode from a U(1) field, is exactly the same as the set of zero modes of the five-brane of type IIA string theory in 10 dimensions investigated in [6].

V. ACKNOWLEDGEMENTS

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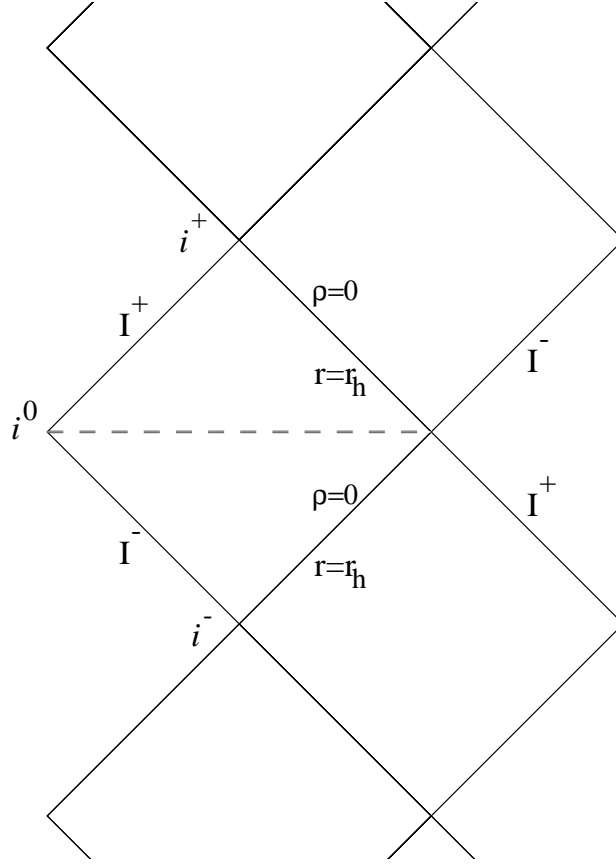[13] H. H. Soleng, *CARTAN: A Mathematica package for tensor analysis*, gr-qc/9502035.

FIG. 1. Causal Structure of the five-brane. Each point represents the product of a four-sphere with a five-plane. The dashed line is a spacelike hypersurface.